Eliminating λ from (12) and (14) by Sylvester's method, the required envelope is given by

$$4(x^2 + y^2 - a^2)^3 - 27a^4y^2 = 0. \tag{15}$$

This is a two-cusped epicycloid, as may be shown by eliminating θ from the two parametric equations of the epicycloid, a and b being the radii of the fixed and generating circles,

$$x = (a + b)\cos\theta - b\cos\frac{a+b}{b}\theta, \tag{16}$$

$$y = (a + b)\sin\theta - b\sin\frac{a+b}{b}\theta, \tag{17}$$

first supposing $b = \frac{1}{2}a$.

Note.—Since making this solution, I discovered that this problem was proposed by Sir Arthur Cayley as No. 1812 in the *London Educational Times*, and in the year 1852.

Cayley was one of the most cordial and active contributors to the mathematical section of the *Times*, and it is interesting to notice that his problems and the most of his solutions appearing in the monthly issues of that journal have been included in his *Works*.

Problem 1812 is reproduced there, but no solution is given. Reference, however, is made to problem 1771 and its solution, his statement being that he was led to No. 1771 by his study of 1812.

The statement of 1771 is: "Given a circle and a line, it is required a find a parabola, having the line for directrix, and the circle for its circle of curvature." The solution given is rather intuitional in character, justifying the equation of the parabola required by certain tests.

Employing rectangular axes, such that $x = m$ is the given line, and $x^2 + y^2 = 1$ the given circle, the required parabola is

$$y^2 - 2\left(1 - \frac{4}{9}m^2\right)^{3/2} y + \frac{16}{27}m^3x + 1 - \frac{4}{3}m^2 = 0 \tag{i}$$

and its focus

$$\left\{m - \frac{8}{27}m^3, \quad \left(1 - \frac{4}{9}m^2\right)^{3/2}\right\}. \tag{ii}$$

Taking $(\cos\theta, \sin\theta)$ as coördinates of a variable point on the circumference of $x^2 + y^2 = 1$, Cayley merely states that

$$x = \frac{3}{2}\cos\theta - m\cos 2\theta + \frac{1}{2}\cos 3\theta,$$

$$y = \frac{3}{2}\sin\theta - m\sin 2\theta + \frac{1}{2}\sin 3\theta,$$

are the coördinates of a variable point on the required envelope, adding what is the interesting connection of Nos. 1812 and 1771, viz., the required envelope in 1812 is a curve of the sixth order and has two cusps, which are the *foci* in the result of solution of 1771. It is to be noticed that the unreduced form of Cayley's result shows that there are *two* parabolas.

By a simple transformation of axes it may be easily shown that the polar of (ii) with respect to (i) is $x = m$, as should be.

The only other place where I have seen our problem is in the American edition of Williamson's *Differential Calculus*, 1884, but there is no certainty about the date when the problem was assigned a place in the manuscript of that text.

Also solved by Paul Capron, S. W. Reaves, H. S. Beers, I. L. Miller, William Weber, A. M. Harding, Horace Olson, G. W. Hartwell, C. P. Sousley, R. A. Johnson, F. M. Morgan, E. W. Worthington, D. F. Barrow, C. C. Yen, and the Proposer.

MECHANICS.

Problem 332 is the same as 490 in Geometry, the solution of which appeared in the February number of the Monthly.

333. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

A flywheel 21 feet in diameter makes 100 revolutions per minute. The weight of a cubic foot of its material is 448 pounds. Show that the intensity of stress on a transverse section of rim,

assuming that it is unaffected by the arms, is 1,176 lbs. per sq. in. If the safe stress permissible in the material is 6,000 lbs. per sq. in., show that the greatest speed at which the wheel can be run with safety is about 225 revolutions per minute.

Solution by William W. Johnson, Cleveland, Ohio.

The unit stress in the flywheel rim due to centrifugal force is found as follows: In the general equation of force, $F = Ma$, the acceleration a in the present instance is v^2/r, in which v = velocity of rim in feet per second, R = radius of rim in feet, W = weight of rim in pounds, and g = acceleration due to gravity. Hence, centrifugal force $= F = Mv^2/R = Wv^2/gR$.

The resultant of half of this force tends to disrupt one half of the rim from the other half. This rupture is resisted by the two sections of the rim at each end of the diameter.

The total tension T in a cross-section of the rim is to one half the sum of half of the radial forces as the diameter of the flywheel is to half its circumference. Therefore,

$$T = F/2\pi = Wv^2/2\pi Rg.$$

Let $W = 2\pi RAw/144$, in which A = area of cross-section of rim in square inches, and w = weight of rim material in pounds per cubic foot.

Then, $T = Awv^2/144g$. Let $T = AS$, where S = stress per unit area of cross-section of rim. Then, $S = wv^2/144g$.[1]

Put $v = \pi Dn/60$, where D = mean diameter of rim in feet, and n = number of revolutions per minute, we have

$$S = \frac{w(\pi Dn)^2}{60^2 \times 144g}. \tag{A}$$

Solving for n, we have

$$n = \frac{720}{\pi D}\sqrt{\frac{Sg}{w}}. \tag{B}$$

Putting, $D = 21$, $n = 100$, $w = 448$, $S = 6,000$, and $g = 32.16$ in formulas (A) and (B) we find

Answer (a): $S = 1,170$ lbs. per sq. in., and

Answer (b): $n = 226$ revolutions per min.

Also solved by J. B. Reynolds.

334. Proposed by HORACE OLSON, Chicago, Illinois.

A particle of elasticity e is projected with velocity v at an angle φ with a plane inclined to the horizontal at an angle ψ; its plane of motion is perpendicular to the inclined plane. Show that after $2v \sin \varphi/g(1 - e) \cos \psi$ seconds it will cease to rebound and will move along the plane with an initial velocity $v \cos \varphi - 2v \sin \varphi \tan \psi/1 - e$ and a uniform acceleration $g \sin \psi$ down the plane.

Solution by Jos. B. Reynolds, Lehigh University.

The acceleration of the particle may be resolved into a component of $g \cos \psi$ perpendicular to the plane and a component $g \sin \psi$ down the plane. Let v_1 be the initial component of the velocity of the particle perpendicular to the plane, v_2 the component of velocity perpendicular to the plane after the first rebound, v_3 after the second rebound, etc. Also, let t_1 be the time from the instant of projection until the first impact, t_2 the time from the first impact until the second, t_3, from the second until the third, etc. Then

$$v_2 = ev_1, \qquad v_3 = ev_2 = e^2 v_1, \quad \text{etc.}$$

$$t_1 = \frac{2v_1}{g \cos \psi}, \qquad t_2 = \frac{2v_2}{g \cos \psi}, \qquad t_3 = \frac{2v_3}{g \cos \psi}, \quad \text{etc.}$$

So that if t is the time before the ball ceases to rebound,

$$t = t_1 + t_2 + t_3 + \cdots = \frac{2v_1}{g \cos \psi}(1 + e + e^2 + \cdots + e^n)_{n=\infty} = \frac{2v_1}{(1 - e)g \cos \psi}$$

$$= \frac{2v \sin \varphi}{(1 - e)g \cos \varphi}, \qquad \text{since} \qquad v_1 = v \sin \varphi.$$

[1] S is independent of the radius and depends only on the rim velocity.